Fundamental Theorem of Calculus FTC II

(a) f integrable on $[a,b]$ $\Rightarrow$ $F(x) = \int_a^x f(t)\,dt$ is continuous

(b) f moreover cont. at $x_0 \in (a,b)$ $\Rightarrow$

F differentiable at x_0

and $F'(x_0) = f(x_0)$

Proof. done in last lecture

For part (b) we showed that for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\Bigg| \underbrace{\frac{F(x) - F(x_0)}{x - x_0}}_{G(x)} - f(x_0) \Bigg| < \varepsilon$$

$$\Rightarrow \lim_{x \to x_0} G(x) = f(x_0) \Leftrightarrow \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

$\Rightarrow$ F differentiable at x_0

Application:

Theorem (Change of Variables)

Let I, J be open intervals

$u: J \to I$ such that u' is continuous
(in particular, u differentiable)

$f: I \to \mathbb{R}$ continuous

$\Rightarrow$ $f \circ u$ is continuous

and
$$\int_a^b f \circ u\,(x)\; u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Proof. Use chain rule for differentiation and FTC I & II.

Fix $c \in I$

let $F(u) = \int_c^u f(t)\,dt$

$\Rightarrow \boxed{F'(u) = f(u)} \quad \forall u \in I$ by FTC II

Let $\boxed{g = F \circ u}$

chain rule $\Rightarrow$ $g'(x) = F'(u(x)) \cdot u'(x)$

$= f(u(x))\, u'(x)$

$$\int_a^b \underbrace{f \circ u(x)\, u'(x)}_{g'(x)}\, dx = \int_a^b g'(x)\,dx \underset{\text{FTC I}}{=} g(b) - g(a)$$

$$= F(u(b)) - F(u(a))$$

$$\underset{\text{def of } F}{\overset{\uparrow}{=}} \int_c^{u(b)} f(t)\,dt - \underbrace{\int_c^{u(a)} f(t)\,dt}_{= -\int_{u(a)}^{c} f(t)\,dt}$$

$$= \int_c^{u(b)} f(t)\,dt + \int_{u(a)}^{c} f(t)\,dt$$

$$= \int_{u(a)}^{u(b)} f(t)\,dt. \qquad (t \to x)$$

rename variable

Example I open interval

Assume $g: I \to \mathbb{R}$ one-to-one and differentiable

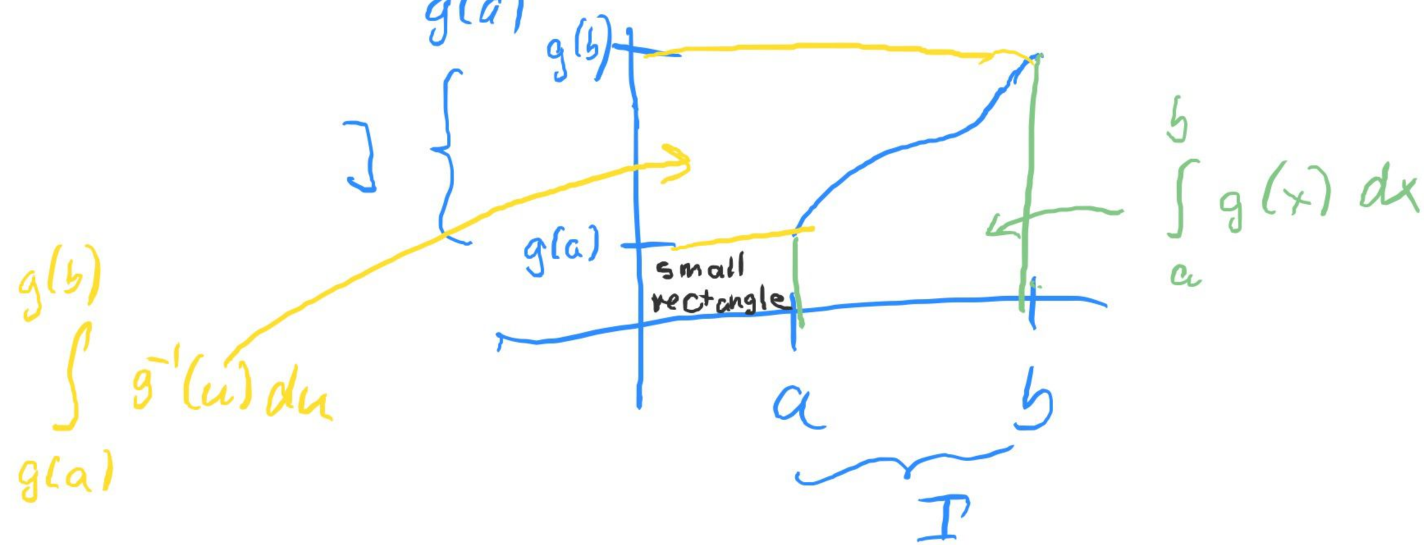
$\rightarrow J = g(I)$ is also an open interval

have shown: $g^{-1}: J \to I$ is also differentiable

claim:

$$\int_a^b g(x)\,dx + \int_{g(a)}^{g(b)} g^{-1}(u)\,du = b\,g(b) - a\,g(a)$$

$(a,b) = I$



Clear from picture:

green area + yellow area = area of big red − area of small red.

i.e. $$\int_a^b g(x)\,dx + \int_{g(a)}^{g(b)} g^{-1}(u)\,du = b\,g(b) - a\,g(a)$$

rigorous proof: put $f = g^{-1}$, $u = g$

apply change of variable formula:

$$\int_a^b \underbrace{g^{-1}\circ g(x)}_{f\circ u(x)}\ \underbrace{g'(x)}_{u'(x)}\,dx = \int_{g(a)}^{g(b)} g^{-1}(u)\,du$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_a^b x\,g'(x)\,dx \underset{\text{integr. by parts}}{=} x\,g(x)\Big|_a^b - \int_a^b g(x)\,dx$$

again: $$\int_{g(a)}^{g(b)} g^{-1}(u)\,du = b\,g(b) - a\,g(a) - \int_a^b g(x)\,dx$$

$\Longrightarrow$ claim

explicit application:

Calculate $\int_0^{1/2} \arcsin x\,dx$ using formula above:

$a=0, \; b=\frac{\pi}{6}$

Solution: Let $g(u) = \sin u$ $\quad 0 \le u \le \frac{\pi}{6}$ $\quad g(a)=0 \quad g(\frac{\pi}{6}) = \frac{1}{2}$

$\Longrightarrow \quad g^{-1}(x) = \arcsin x \quad 0 \le x \le 1/2$

by formula: $$\int_0^{1/2} \arcsin x\,dx + \int_0^{\pi/6} \sin u\,du = \frac{\pi}{6}\cdot\frac{1}{2} - 0\cdot 0$$

Solve for first integral:

$$\int_0^{1/2} \arcsin x \, dx = \frac{\pi}{12} - \int_0^{\pi/6} \sin x \, dx$$

$$= \frac{\pi}{12} - \left(-\cos x \Big|_0^{\pi/6} \right)$$

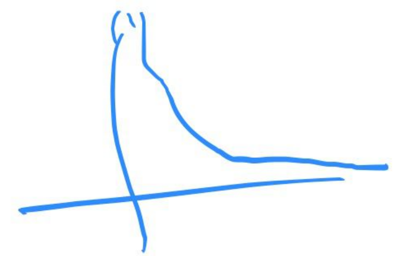
$$= \frac{\pi}{12} + \cos\frac{\pi}{6} - \cos 0$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Remarks:

① We have defined integrals only for bounded functions over a finite interval.
In some cases, this can be extended to unbounded functions and/or intervals of infinite length.

Example. $f(x) = \frac{1}{\sqrt{x}}$



$f(x)$ unbounded on interval $[0,1]$

but bounded on interval $[\varepsilon, 1]$, $\varepsilon > 0$, $\varepsilon < 1$

$$\int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx = \int_{\varepsilon}^{1} x^{-1/2}\,dx = \frac{x^{1/2}}{1/2}\Big|_{\varepsilon}^{1} = 2 - 2\sqrt{\varepsilon}$$

In this case, we can define

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx =$$

$$= \lim_{\varepsilon \to 0} 2 - 2\sqrt{\varepsilon} = 2$$

Similarly, we can define

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{K \to \infty} \int_1^K x^{-2}\,dx = \lim_{K \to \infty} -x^{-1}\Big|_1^K$$

$$= \lim_{K \to \infty} 1 - \frac{1}{K} = 1$$

These types of integrals, defined via limits in integration limits are called improper integrals

(2) have seen that the function $f: [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

is not integrable in our definition via Darboux sums.

There exists a more general approach, called the Lebesgue integral, which agrees with our definition for functions for which the integral is defined, but which also works for more general functions like the one above.

The value of the Lebesgue integral for f as above is equal to 0.